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LOCAL BOUNDEDNESS OF MONOTONE OPERATORS UNDER MINIMAL HYPOTHESES

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We give a short proof the local boundedness of a monotone operator as an easy consequence of the continuity of an associated convex function.

Let X be a real normed space with normed dual X^*. Recall that a set-valued mapping $T\colon X \to X^*$ is *monotone* if

$$\langle x^* - y^*, x - y\rangle \geqslant 0$$

for any pairs $x^* \in T(x)$, $y^* \in T(y)$. The *domain* of T, $D(T)$, is defined by $D(T) := \{x \in X : T(x) \neq \emptyset\}$. The mapping is *locally bounded* at $x \in D(T)$ if for some $\varepsilon > 0$ the set $T(\{y \in X : \|y - x\| < \varepsilon\})$ is bounded. For any set S in X, the cone generated by S at x is $S_x = \{y \in X \mid y = t(s - x),\, s \in S,\, t > 0\}$. We will (a little non-standardly) call a point x of S an *absorbing point* of S if $S_x = X$. Clearly every interior point is an absorbing point. If S is convex then absorbing points and core points coincide.

A function $f\colon X \to\,]-\infty, \infty\,]$ is *convex* if $\operatorname{epi}(f) = \{(x, r) \in X \times R \mid f(x) \leqslant r\}$ is a convex set, and *lower semicontinuous* if $\{x \in X \mid f(x) \leqslant r\}$ is closed for each real r. [Here X may be any real topological vector space.] The *essential domain* of f is $\operatorname{dom}(f) = \{x \in X : f(x) < \infty\}$. Finally, recall that a real locally convex space X is *barrelled* if each closed convex set C in X for which 0 is an absorbing point [that is $C_0 = X$] is a neighbourhood of zero. It follows from the Baire category theorem that every Banach space is barrelled, as indeed is every Baire normed space. Moreover, a Banach space may possess many non-Baire but barrelled normed subspaces including some dense hyperplanes [**1**]. The following result is basic to much convex analysis [**2**]. We include a proof of a somewhat more general version.

THEOREM 1. *Let X be a barrelled locally convex space. Let $f\colon X \to\,]-\infty, \infty\,]$ be convex and lower semicontinuous and suppose that x is an absorbing point of $\operatorname{dom}(f)$. Then f is continuous at x.*

PROOF: We may assume by translation that $f(0) = 0$ and that 0 is an absorbing point of $\operatorname{dom}(f)$. Let $\varepsilon > 0$ be given and set $C = \{x \in X \mid f(x) \leqslant \varepsilon\}$. Since f is

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convex and lower semicontinuous, C is convex and closed. We verify that $U = C \cap -C$ is a symmetric neighbourhood of zero. As X is barrelled it suffices to show that 0 is an absorbing point of C. Let $x \in X$. By hypothesis $f(rx) < \infty$ for some $r > 0$. Select $\lambda > 0$ such that $\lambda f(rx) < \varepsilon$ and $\lambda < 1$. Then λrx lies in C because

$$f(\lambda rx) \leqslant \lambda f(rx) + (1-\lambda) f(0) = \lambda f(rx) < \varepsilon.$$

Finally, since $f(-x) + f(x) \geqslant 0$, we have $|f(x) - f(0)| \leqslant \varepsilon$ for all x in U. ∎

We may now establish our main result.

THEOREM 2. *Let X be a barrelled normed space. Let $T\colon X \to X^*$ be monotone and suppose that x is an absorbing point of $D(T)$. Then T is locally bounded at x.*

PROOF: We may assume by translation that $0 \in T(0)$ and that 0 is an absorbing point of $D(T)$ (which need not be convex). Consider the function $f\colon X \to]-\infty, \infty]$ given by

$$f(x) = \sup\{\langle y^*, x - y\rangle : y^* \in T(y), \|y\| \leqslant 1\}.$$

Indeed, f is always nonnegative since $0 \in T(0)$. Then f is convex and lower semicontinuous as a supremum of continuous affine functionals. Let $x \in X$. As 0 is an absorbing point of $D(T)$, we can find $t > 0$ with $T(tx) \neq \emptyset$. Choose any $u^*(t) \in T(tx)$. For each $y^* \in T(y)$

$$\langle y^*, tx - y\rangle \leqslant \langle u^*(t), tx - y\rangle$$

because T is monotone. Since $0 \in T(0)$ this also shows $f(0) = 0$. Hence

$$f(tx) \leqslant \sup\{\langle u^*(t), tx - y\rangle :\ \|y\| \leqslant 1\} < \infty.$$

This shows that 0 is an absorbing point of $\operatorname{dom}(f)$. Thus Theorem 1 ensures that for some $\delta > 0$, $f(x) \leqslant 1$ whenever $\|x\| \leqslant 2\delta < 1$.

Equivalently, if $y^* \in T(y)$, $\|y\| \leqslant 1$, $\|x\| \leqslant 2\delta$, then

$$\langle y^*, x\rangle \leqslant \langle y^*, y\rangle + 1.$$

Hence if $y^* \in T(y)$ with $\|y\| \leqslant \delta$, then

$$2\delta \|y^*\| = \sup\{\langle y^*, x\rangle :\ \|x\| \leqslant 2\delta\} \leqslant \|y^*\|\,\|y\| + 1 \leqslant \delta\,\|y^*\| + 1$$

and $\|y^*\| \leqslant 1/\delta$. This establishes that T is locally bounded at 0. ∎

Remarks. 1. The usual result is established under the stronger assumptions that x is an interior point of $D(T)$, and that X is Banach or Baire [**4**, **3**]. In our setting this is not presumed. For instance, $D = \{(x, y) : x^2 + y^2 = 2|x|\} \cup \{(0, y) :\ |y| = 1\}$ has 0 as an absorbing point and so is a possible domain in $\mathbb{R}^2$.

2. We could establish directly that $\{x \mid f(x) \leqslant 1\}$ is a barrel and so omit Theorem 1. This would also omit the pleasant parallel between the statements of the two results.

3. This technique also works of $x \in$ core conv $D(T)$, as in [**4**].

References

[1] J.M. Borwein and D.A. Tingley, 'On supportless convex sets', *Proc. Amer. Math. Soc.* **94** (1985), 471–476.

[2] R.B. Holmes, *Geometric Functional Analysis and Applications* (Springer-Verlag, New York, 1975).

[3] R.R. Phelps, *Convex functions, Monotone Operators and Differentiability*, Lecture Notes in Mathematics (Springer-Verlag, University of Washington, 1989). (to appear) .

[4] R.T. Rockafellar, 'Local boundedness of nonlinear monotone operators', *Mitchigan Math. J.* **16** (1969), 397–407.

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